1. Inversions in Permutations

An *inversion* in a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i, j) of indices such that $i < j$ but $\sigma(i) > \sigma(j)$. Let $\mathrm{inv}(\sigma)$ denote the total number of inversions of σ. One can uniquely record a permutation $\sigma \in \mathfrak{S}_n$ by recording, for each $1 \leq i \leq n$ the number of inversions (i, j), which will be a number from 0 to $n-i$. One can recover the i^{th} entry of the permutation (and hence the whole permutation) from such a recording by the following algorithm. Let S be the set of the first $i-1$ entries of the permutation. If the number of inversions of the form (i, j) is k, then there must be exactly k elements after the i^{th} position which are smaller than it. Hence, the i^{th} entry of the permutation is the $(k+1)^{\text{th}}$ smallest element of $\{1, \ldots, n\} \setminus S$.

Hence,

$$(\mathfrak{S}_n, \mathrm{inv}) \xrightarrow{\sim} \left(\prod_{i=1}^{n} \{0, \ldots, n-i\}, \omega \right)$$

is a weight preserving bijection, where ω is the usual additive weight.

By simply reversing the sequences (using the index substitution $j = n - i$), we get a further bijection:

$$(\mathfrak{S}_n, \mathrm{inv}) \xrightarrow{\sim} \left(\prod_{j=0}^{n-1} \{0, \ldots, j\}, \omega \right)$$

So these two sets will have the same generating series. Using the product and sum lemma, we obtain:

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} &= \prod_{i=0}^{n-1} \sum_{k=0}^{i} q^k = \prod_{i=0}^{n-1} \frac{1 - q^{i+1}}{1 - q} \\ &= \frac{\prod_{i=0}^{n-1} (1 - q^{i+1})}{(1-q)^n} = \frac{(q;q)_n}{(1-q)^n} \\ &= [n]_q! \end{aligned}$$

2. Inversions in Words

We now extend the definition of inversions to words on the alphabet $\{1, \ldots, k\}$. If $w(i)$ is the symbol at the i^{th} position of the word w, then an inversion in w is a pair (i, j) of positions such that $i < j$, but $w(i) > w(j)$.

We wish to know the generating series with respect to inversions for the set $\mathfrak{W}^{(n)}_{n_1, \ldots, n_k}$ of words of length n on $\{1, \ldots, k\}$ with n_i occurrences of i.

To get it, we will use an indirect decomposition, decomposing an element $\sigma \in \mathfrak{S}_n$ as a word $w \in \mathfrak{W}^{(n)}_{n_1, \ldots, n_k}$, together with permutations $\sigma_1 \in \mathfrak{S}_{B_1}, \ldots, \sigma_k \in \mathfrak{S}_{B_k}$ in a bijection which additively preserves inversions, where $n = n_1 + n_2 + \cdots + n_k$ and B_i is a set with n_i elements.

That is, our weight-preserving bijection will look like:

$$(\mathfrak{S}_n, \mathrm{inv}) \xrightarrow{\sim} (\mathfrak{W}^{(n)}_{n_1, \ldots, n_k}, \mathrm{inv}) \times (\mathfrak{S}_{B_1}, \mathrm{inv}) \times \cdots \times (\mathfrak{S}_{B_k}, \mathrm{inv})$$

We split the domain of a permutation $\sigma \in \mathfrak{S}_n$ into blocks with sizes $n_1, \ldots, n_k$ in the obvious way, with the i^{th} block consisting of the elements $B_i = \{m_i + 1, \ldots, m_i + n_i\}$ where $m_i = n_1 + \cdots + n_{i-1}$. Note that $\mathfrak{S}_{n_i}$ acts on B_i in a canonical way, with a permutation $\rho \in \mathfrak{S}_{n_i}$ sending $m_i + j$ to $m_i + \rho(j)$.

The word w is then defined by $w(j) = i$ if $\sigma(j) \in B_i$.

We define the permutation σ_i for each $i \in \{1, \dots, k\}$ as the subword of σ consisting of the elements of B_i. Note that if u and v are both elements of B_i, then (u, v) is an inversion of σ if and only if it is also is an inversion of σ_i.

It will help to illustrate this bijection with an example. Take $n_1 = 3$, $n_2 = 3$ and $n_3 = 2$, so that $B_1 = \{1, 2, 3\}$, $B_2 = \{4, 5, 6\}$ and $B_3 = \{7, 8\}$.

Let $\sigma \in \mathfrak{S}_8$ be the permutation (in word form):

$$\sigma = \begin{pmatrix} 6 & 2 & 5 & 3 & 1 & 4 & 8 & 7 \end{pmatrix}$$

Then we construct the word $w \in \mathfrak{W}^{(8)}_{3,3,2}$ by taking $w(i)$ to be the block in which $\sigma(i)$ occurs:

$$w = \begin{pmatrix} 2 & 1 & 2 & 1 & 1 & 2 & 3 & 3 \end{pmatrix}$$

We then compute:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 6 & 5 & 4 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 8 & 7 \end{pmatrix} \end{aligned}$$

To invert the above process, we simply replace the j^{th} occurrence of k in the word w with $\sigma_k(m_k + j)$. Hence, this is a bijection.

To see that it is additively weight-preserving with respect to inversions, we do a bit of a case analysis. We classify the inversions (u, v) of σ according to whether $\sigma(u)$ and $\sigma(v)$ are in the same block B_i or in different blocks.

If they are in different blocks, say $\sigma(u) \in B_i$ and $\sigma(v) \in B_j$, then since $\sigma(u) > \sigma(v)$, we must have $i > j$, and so $w(u) = i > j = w(v)$. Hence (u, v) is also an inversion of w. Conversely, every inversion of w will clearly be an inversion of σ of this type, since if $u < v$ but $i = w(u) > w(v) = j$, we have that $\sigma(u) \in B_i$ and $\sigma(v) \in B_j$, but since $i > j$, any element of B_i is greater than any element of B_j, and in particular $\sigma(u) > \sigma(v)$.

If they are in the same block B_i, then by a previous argument, (u, v) is an inversion of σ if and only if it is an inversion of σ_i.

Hence, we have that $\operatorname{inv}(\sigma) = \operatorname{inv}(w) + \sum_{i=1}^{k} \operatorname{inv}(\sigma_i)$.

By subtracting m_i from each of the elements of σ_i, we obtain a standard permutation $\sigma_i^* \in \mathfrak{S}_{n_i}$, and this is obviously inversion-preserving.

So, we have a weight-preserving bijection:

$$\begin{aligned} (\mathfrak{S}_n, \operatorname{inv}) &\xrightarrow{\sim} (\mathfrak{W}^{(n)}_{n_1,\dots,n_k}, \operatorname{inv}) \times (\mathfrak{S}_{B_1}, \operatorname{inv}) \times \cdots \times (\mathfrak{S}_{B_k}, \operatorname{inv}) \\ &\xrightarrow{\sim} (\mathfrak{W}^{(n)}_{n_1,\dots,n_k}, \operatorname{inv}) \times (\mathfrak{S}_{n_1}, \operatorname{inv}) \times \cdots \times (\mathfrak{S}_{n_k}, \operatorname{inv}) \end{aligned}$$

Hence, we have a corresponding equation of generating series. Let $W^{(n)}_{n_1,n_2,\cdots,n_k}(q)$ be the generating series for $(\mathfrak{W}^{(n)}_{n_1,\dots,n_k}, \operatorname{inv})$. Then:

$$[n]_q! = W^{(n)}_{n_1,n_2,\cdots,n_k}(q)[n_1]_q![n_2]_q! \cdots [n_k]_q!$$

and hence,

$$\begin{aligned} W^{(n)}_{n_1,n_2,\cdots,n_k}(q) &= \frac{[n]_q!}{[n_1]_q![n_2]_q! \cdots [n_k]_q!} \\ &= \binom{n}{n_1, n_2, \dots, n_k}_q \end{aligned}$$