1. Transitive Permutation Groups - 22/11/09

This post is a follow on from the previous post on permutation groups. We need a few definitions.

Definition 1.1 (Transitive Permutation Group)**.** *A permutation group* (G, Ω) *is said to be* transitive *if for any* $\alpha, \beta \in \Omega$ *there exist* $g \in G$ *such that* $\alpha^g = \beta$

All the permutation groups we have considered so far have been transitive.

Definition 1.2 (Orbit)**.** *If* (G, Ω) *is a permutation group and* $\alpha \in \Omega$ *then the* orbit of α under the action of G *is:*

$$\alpha^G = \{\alpha^x : x \in G\}$$

For a transitive permutation group $\alpha^G = \Omega$. Otherwise Ω maybe be partitioned into orbits: $\Omega = \coprod_{i=1..n} \Omega_i$ such that for each i, the permutation group (G, Ω_i) is transitive.

For an example of a non-transitive permutation group, consider the way a group acts on itself by conjugation.

$$x^g = g^{-1}xg$$

The orbits of this action are the conjugacy classes of the group. Now for some more definitions:

Definition 1.3 (Stabilizer)**.** *Let* (G, Ω) *be a transitive permutation group, and let* α *be any point of* Ω*. The* point stabilizer *of* α *is:*

$$G_\alpha = \{x \in G : \alpha^x = \alpha\}$$

In the example of the group acting on itself by conjugation, the stabilizer of a point g is precisely those x which commute with g.

The following fact is crucial:

$$\boxed{G_{\alpha^g} = g^{-1}G_\alpha g \qquad \text{for any } g \in G}$$

To see what this is true, suppose that $x \in G$ is such that:

$$(\alpha^g)^x = \alpha^g$$

It follows that $\alpha^{gxg^{-1}} = \alpha$ and so $gxg^{-1} \in G_\alpha$. That is:

$$x \in g^{-1}G_\alpha g$$

Conversely suppose that $x \in g^{-1}G_\alpha g$. Then $gxg^{-1} \in G_\alpha$ and so $\alpha^{gxg^{-1}} = \alpha$. That is $\alpha^{gx} = \alpha$, so $x \in G_{\alpha^g}$.

Since (G, Ω) is transitive, every element of Ω may be expressed in the form α^g for some $g \in G$. Thus we see that the point stabilizers of a transitive representation of G form a family of conjugate subgroups of G.

From here it is relatively straightforeward to see that every transitive permutation group is permutation equivalent to one of the form G acting on the cosets of some subgroup H.

Let (G, Ω) be a transitive permutation group and α is some point of Ω. Let $\cos_G(G_\alpha)$ denote the set of right cosets of the point stabilizer G_α For each $\beta \in G$ Consider:

$$S_\beta = \{x \in G : \alpha^x = \beta\}$$

If $\beta = \alpha^g$ then it's not hard to see that S_β is a right coset of G_α:

$$S_\beta = G_\alpha g$$

Define $\eta : \Omega \to \cos_G(G_\alpha)$ by $\eta(\beta) = S_\beta$ and take $\varphi : G \to G$ to be the identity. We have, for any $\beta \in \Omega$ and any $x \in G$, if $\beta = \alpha^g$ then:

$$\begin{aligned}
\eta(\beta^x) &= S_{\beta^x} \\
&= S_{\alpha^{gx}} \\
&= G_\alpha gx \\
&= G_\alpha g^{\varphi(x)} \\
&= \eta(\alpha^g)^{\varphi(x)} \\
&= \eta(\beta)^{\varphi(x)}
\end{aligned}$$

Thus η and φ establish the desired permutation equivalence.

As a consequence of this, we have that for any $g \in G$ the size of the conjugacy class of g in G is given by the size of the group divided by the number of elements of G which commute with g.

(1) Definition of a transitive permutation group
(2) Definition an orbit
(3) Definition of a point stabilizer
(4) Classification of transitive permutation groups
(5) Formula for the size of an orbit